Georg Cantor – The 19^{th} Century Tackles Infinity

Major Source: Journey Through Genius, William Dunham, chapter 11 and 12 (1990). A highly recommended book.

Suggestion: Don’t read this without paper and pencil handy!

The 19^{th} century marked the ‘formalization’ of mathematics. Partly the reason was to ‘firm up’ all of mathematics to create a logical structure for it (like Euclid pretty much did with Geometry and some number theory in **Elements**), but another reason was to create the foundation to address some nagging details left over from the tremendous growth of knowledge in the 18^{th} century. The epsilon-delta definition of Karl Weierstrass (Anton 2.4) is a product of this effort.

Most of you will accept that between any two rational numbers (expressible as a ratio of two integers – i.e. fractions!) there are an infinite number of irrational (and rational) numbers. By ‘infinite number’ we mean bigger than any finite number you can name. Similarly you will accept that between any two irrational numbers (**not** expressible as fractions!) there are an infinite number of rational (and irrational) numbers.

So you would conclude that the set of **all** real numbers are somewhat equally divided between rational and irrational numbers.

Surprisingly you would be wrong – incredibly, unbelievably, flamingly wrong.
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Let’s consider a really finite set – the number of fingers on each hand (assuming five each for the sake of my argument). There are a zillion ways of determining that you have the same number of fingers on each hand. We’re going to consider just one: match one pinky with the other; one thumb with the other; one ring finger … and so on. This “matching” process is called a **one-to-one correspondence**. If there is a 1-1 correspondence between the fingers of one hand to the fingers of another, then you have the same number of fingers. We say the **cardinality** of those sets is the same (and equals 5).

We will use **EXACTLY** that process to decide if two infinite sets are ‘the same size’ (have the same cardinality). Two sets, finite or infinite, are equivalent, or have the same cardinality, if there exists a one-to-one correspondence between the elements of the first set and the elements of the second set.

“So far, so good” I hear you mutter …. But the trouble has only just begun!
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Look at the one-to-one correspondence that sends 1 => 2, 2 => 4, 3 => 6, 4 => 8, and so on. Every even number will have something mapped to it. Every natural number will be mapped to an even number. And so the two sets are equivalent.

We have annihilated all the odd numbers in $\mathbf{N}$ (the set of natural numbers) and have the same number of elements left!
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OK – most of you aren’t alarmed yet. HAHAHA! It gets better (did I mention that Cantor spent some time suffering from diminished sanity or depression?)

To continue, Cantor called infinite sets that could be put into a one-to-one correspondence with $\mathbb{N}$ **denumerable sets** or **countably infinite sets.** The size of a denumerable set he denoted by the symbol $\aleph_0$. It is pronounced “aleph null” – aleph being the first letter in Hebrew. It is the first ‘**transfinite**’ cardinal.

Next we consider a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Q}$ (the set of all rational numbers – think ‘Q’ for ‘quotient’).

0
1 -1 2 -2 3 -3 ….
1/2. -1/2 2/2 -2/2 ….
1/3 -1/3 2/3 -2/3 …
¼ -1/4 ….
and so on. Each row has the same denominator, and the numerator in each row alternates signs and increases in absolute value.

Traverse the ‘table’ in this order:
0 ~ 1 ~ ½ ~ -1 ~ 2 ~ -1/2 ~ 1/3 ~ ¼ ~ -1/3 ~ ….
Do you see what’s happening? (Try to draw it out with everything lined up nicely – NOT like I can do it by typing here ….). Start at zero, go down 1, then up diagonally, then over one, then down diagonally, then down one, and so on….

Once you’re “done”, you can probably see that a one-to-one correspondence has been created between $\mathbb{N}$ and $\mathbb{Q}$. If you aren’t completely clear here, don’t worry about it – I have my days where I’m just not sure myself …. But then you have to trust us. There is no “formula” that I know of, but you can see that each fraction will have been selected (some more than once!), and each natural number used.

So – what’s the deal? Is every infinite set denumerable? That would certainly be nice, but who would go crazy if that were the case?

The **counter-example** is the open interval of all real numbers between 0 and 1 … (0,1). This special interval – in this context is called **the continuum**. I know – it sounds like we’re in some Star Trek series. But proving that there is **NOT** a one-to-one correspondence is neat!

Let’s assume there **is** a one-to-one correspondence between $\mathbb{N}$ and (0, 1). For simplicity’s sake, we will insist that each decimal representation appear only once (in other words, we’ll choose .2000000 instead of .1999999…. (and if you’ve never seen THAT trick, let me know!). Then, every decimal in (0, 1) is matched uniquely with a counting number. We will then create a *new* number like this:

* pick a digit from (1, 2, 3, 4, 5, 6, 7, or 8) that is different from the first digit in the first number.
  that will be the first digit in our new decimal representation.
* pick a digit from 1 … 8 that is different from the second digit of the second number.
  that will be the second digit in our new decimal representation.
* and continue ….

When you are ‘done’ you will have a decimal representation of a number in (0, 1) that is different from every single number in the one-to-one correspondence. Therefore, **there IS no one-to-one correspondence** and so the cardinality of (0, 1) is larger than $\aleph_0$. We’ll denote that by **c** – the cardinality of the continuum.

A couple of more 'facts' so you get your money's worth ….

The set of "algebraic" numbers (those numbers that are roots of polynomials with integer coefficients) is denumerable.

The set of irrational numbers that are not algebraic is called the set of transcendental numbers. Its cardinality is **c.** π and e are transcendental numbers.

Determining whether a specific number is transcendental is very difficult. For example, the number e was proven to be transcendental in 1873. π was proven to be transcendental in 1882. See Mathworld for a list of numbers we know to be transcendental.

The union of two denumerable sets is … denumerable.

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Let's summarize (and I can't imagine what I could assign as extra credit here ….).

There are different 'sizes' of infinite sets.

$\mathbb{R}$ (set of all real numbers) is not denumerable.

$\mathbb{R}$ consists of $\mathbb{Q}$ – the rational numbers which are denumerable, and
the set of irrational numbers, which is not denumerable.

The set of irrationals consists of the set of algebraic numbers, which is denumerable, and
the set of transcendental numbers, which is not denumerable.

The sets $\mathbb{N}$ (natural numbers), $\mathbb{W}$ (whole numbers … $\mathbb{N}$ + 0), and $\mathbb{Z}$ (the set of integers – from the German Zahl) are all denumerable.

So … you have a microscopic flea standing on only 1 point on the number line. He jumps and randomly lands on another point. What is the probability that he will land on a transcendental number? …

100%.

The numbers that we study in school … even in the 'hard' problems … are very unlike the typical number in existence.

And that's a sobering fact ….

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Extra Credit: Ok I changed my mind. Research and write a short essay on what comes after $\aleph_0$.

Other sources:

http://mathworld.wolfram.com/TranscendentalNumber.html

And of course let me know if you find other good ones.